A SOLVABLE MODEL OF WEAK TURBULENCE

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A model of weak turbulence amenable to solution is investigated. There are three modes of interacting waves whose laws of dispersion are so chosen that it is possible to go over to the diffusion approximation in k-space. The steady-state spectrum of the turbulence in the presence of regions of instability, transparency and damping is found. The formal apparatus of nonlinear wave dynamics is also discussed.

In the investigation of plasma turbulence cases are encountered where the turbulent state is a system of intense interacting oscillations. If it is assumed that the amplitudes of the oscillations are not too large and the random phase hypothesis is adopted, this state may be described with the aid of the kinetic equation [1-4]. In these circumstances it is natural to speak of weak plasma turbulence.

Mathematically, the kinetic equations are very hard to solve, so that it is customary to make do with an estimate. A similar situation exists in connection with the theory of gravitational waves at the surface of a liquid. Accordingly, it is convenient to talk in terms of general nonlinear wave dynamics, i. e., to consider an arbitrary medium in which waves subject to nonlinear interaction can propagate.

Within the framework of nonlinear wave dynamics it is possible to construct models for which the kinetic equations can be solved. The study of one such model is the subject of this article. The possibility of solving the model exists because in a given case we can go over to the diffusion approximation in k-space and solve the differential equations obtained. We can also get the steady-state spectrum of the turbulence in the presence of a region of instability and a region of damping.

In this article the author applies the apparatus of nonlinear wave dynamics, which was previously employed, at least in part, in [1, 5]. Since this apparatus is not generally known, it will first be described in some detail.

1. Formalism of nonlinear wave dynamics. The essence of the method consists in going over to new variables – complex wave amplitudes that are classical analogues of the quantum operators of particle production and annihilation. The transition has a parallel in the method of variation of arbitrary constants in the theory of differential equations.

Consider wave propagation in an infinite homogeneous medium. Let the medium be described by a set of n real variables $\chi_1, \ldots, \chi_n$, dependent on time and the coordinates, and let these variables satisfy an equation containing a linear and a bilinear part and which is invariant with respect to space-time displacements.

In its most general form this equation may be written:

$$\begin{gathered}\int^{t} dt_1 \int d\mathbf{r}_1 G_{nm}(t-t_1, \mathbf{r}-\mathbf{r}_1)\, \chi_m(t_1, \mathbf{r}_1)\, dt_1\, d\mathbf{r}_1 = \\ = \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 \int d\mathbf{r}_1 \int d\mathbf{r}_2 L_{nml}(t-t_1,\ t-t_2,\ \mathbf{r}-\mathbf{r}_1,\ \mathbf{r}-\mathbf{r}_2)\, \chi_m(t_1,\mathbf{r}_1)\ \chi_l(t_2,\mathbf{r}_2)\end{gathered} \tag{1.1}$$

where G_{nm} and L_{nml} are real coefficient functions. If Eqs. (1.1) are differential, they contain derivatives of the δ-functions.

After a Fourier transformation with respect to time and the coordinates, we get:

$$\begin{gathered}G_{mn}(\mathbf{k}, \omega)\, \chi_m(k, \omega) = \int d\omega_1 \int d\omega_2 \int d\mathbf{k}_1 \int d\mathbf{k}_2 L_{nml}(\omega, \omega_1, \omega_2, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \times \\ \times \delta(\omega-\omega_1-\omega_2)\, \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\, \chi_m(\mathbf{k}_1, \omega_1)\, \chi_l(\mathbf{k}_2, \omega_2)\,.\end{gathered} \tag{1.2}$$

Since the starting functions are real, the Hermite conditions

$$\begin{gathered}\chi_n^+(k, \omega) = \chi_n^*(-\mathbf{k}, -\omega) = \chi_n(\mathbf{k}, \omega), \quad G_{nm}^+(\mathbf{k}, \omega) = G_{nm}(\mathbf{k}, \omega) \\ L_{nml}^+(\omega, \omega_1, \omega_2, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = L_{nml}(\omega, \omega_1, \omega_2, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\end{gathered} \tag{1.3}$$

are fulfilled. We shall assume that ω is complex. Then G and L possess definite analytic properties with respect to the frequency variables.

If it is to be meaningful to speak of linearization, the function $L_{nml}(\omega, \omega_1, \omega_2, k, k_1, k_2)$ must not become infinite for real values of the arguments. The linearized equation has the form:

$$G_{nm}(\mathbf{k},\omega)\chi_m(\mathbf{k},\omega)=0 \quad .$$

It has a solution for certain values of $\omega_s(k)$ determined from the condition

$$\det|G_{nm}|=0 \quad . \tag{1.4}$$

We shall make the following assumptions concerning the zeros of Eq. (1.4).

1) The number of zeros is finite, which corresponds to a finite number of wave modes capable of propagating in the medium.

2) The zeros enter into Eq. (1.4) in pairs, so that positive and negative subscripts can be attached to them, and

$$\omega_{\pm s}(\mathbf{k})=\pm\omega_{0s}(\mathbf{k})+i\nu(\mathbf{k}) \quad . \tag{1.5}$$

The real part of the frequency can be made positive. This condition is the result of symmetry with respect to reflection of the coordinates and holds for a medium at rest.

3) The function $\omega_{0s}(k)\not\equiv 0$. This condition enables us to exclude nonwavelike motions of the medium. In real problems, in hydrodynamics, for example, such motions may be present, in which case they must be considered separately.

We shall also disregard the case of multiple roots of Eq. (1.4); this arises, for example, in connection with waves of different polarization in a nongyrotropic medium. In principle, this case is not any more difficult, but it does lead to more complicated calculations.

The general Hermitian solution of Eq. (1.4) is

$$\begin{gathered}\chi_n(\mathbf{k},\omega)=\sum_{s=-p}^{p}A_{ns}(\mathbf{k})\,a_s(\mathbf{k})\,\delta(\omega-\omega_s(\mathbf{k})),\quad A_{n,-s}(\mathbf{k})=A_{ns}^{+}(\mathbf{k})\\ a_{-s}(\mathbf{k})=a_s^{+}(\mathbf{k}) \quad .\end{gathered} \tag{1.6}$$

We shall seek a solution of nonlinear equation (1.2) in the form:

$$\chi_n(\mathbf{k},\omega)=\sum_{s=-p}^{p}A_{ns}(\mathbf{k})\,a_s(\mathbf{k},\omega) \tag{1.7}$$

where $a_s(k,\ \omega)$ are the new variables. We substitute (1.7) in (1.2). The left side of Eq. (1.2) assumes the form:

$$G_{nm}(\mathbf{k},\omega)\,A_{ms}(\mathbf{k})\,a_s(\mathbf{k},\omega)=G_{ns}^{(1)}(\mathbf{k},\omega)\,(\omega-\omega_s(\mathbf{k}))\,a_s(\mathbf{k},\omega). \tag{1.8}$$

The rank r of the matrix $G_{ns}^{(1)}$ is equal to the rank of the matrix A_{ns}, and, obviously, can not be greater than 2p. If r = 2p, then the transformed equation can at once be solved with respect to $(\omega-\omega_s(k))a_s(k,\ \omega)$. If r < 2p, it is necessary to impose 2p − r additional conditions on a_s. We choose these conditions in the form:

$$\sum_{s=-p}^{p}(\omega-\omega_s(\mathbf{k}))^{\alpha}a_s(\mathbf{k})=0 \qquad (\alpha=1,\ldots,2p-r). \tag{1.9}$$

Obviously, they are independent of each other and can not be expressed linearly in terms of $A_{ns}(k)$. Equations (1.8) and (1.9) can be solved with respect to $(\omega-\omega_s(k))a_s(k,\ \omega)$. As a result we get:

$$\begin{gathered}(\omega-\omega_s(\mathbf{k}))\,a_s=-\int d\omega_1\int d\omega_2\int d\mathbf{k}_1\int d\mathbf{k}_2\,M_{ss_1s_2}(\omega,\ \omega_1,\ \omega_2,\ \mathbf{k},\ \mathbf{k}_1,\ \mathbf{k}_2)\times\\ \times\,\delta(\omega-\omega_1-\omega_2)\,\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,a_{s_1}(\mathbf{k}_1,\omega_1)\,a_{s_2}(\mathbf{k}_2,\omega_2)\end{gathered} \tag{1.10}$$

where $M_{ss_1s_2}$ is the transformed function L_{nml}. After an inverse Fourier transformation with respect to time we get:

$$\begin{gathered}\left(i\frac{\partial}{\partial t}-\omega_s(\mathbf{k})\right)a_s(\mathbf{k},t)=\int_{-\infty}^{t}dt_1\int_{-\infty}^{t}dt_2M_{ss_1s_2}(t-t_1,\ t-t_2,\ \mathbf{k},\ \mathbf{k}_1,\ \mathbf{k}_2)\times\\ \times\,\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,a_{s_1}(t_1,\mathbf{k}_1)\,a_{s_2}(t_2,\mathbf{k}_2)\,d\mathbf{k}_1d\mathbf{k}_2 \,.\end{gathered} \tag{1.11}$$

If the amplitudes are small enough, Eq. (1.11) can be simplified. We shall make the change of variables

$$a_s(\mathbf{k},t)=c_s(\mathbf{k},t)\,e^{-i\omega_s(\mathbf{k})t} \,. \tag{1.12}$$

Substituting (1.12) in (1.11), we get:

$$i\frac{\partial c_s}{\partial t}=\int d\mathbf{k}_1\int d\mathbf{k}_2\exp\{it[\omega_s(\mathbf{k})-\omega s_1(\mathbf{k}_1)-\omega s_2(\mathbf{k}_2)]\}\int_0^{\infty}d\tau_1\int_0^{\infty}d\tau_2\times$$

$$\times M_{ss_1s_2}(\tau_1, \tau_2, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\, c_{s1}(t-\tau_s, \mathbf{k}_1)\, c_{s2}(t-\tau, \mathbf{k}_2) \exp\{i[\omega_{s_1}(\mathbf{k}_1\tau_1+\omega_{s_2}(\mathbf{k}_2)\tau_2]\}\,.$$

If the c_s are sufficiently small, we can neglect the derivatives of c_s with respect to time, as in the Bogolyubov-Krylov method, and take c_{s_1} and c_{s_2} out of the integrals with respect to τ. Reverting to the variables a_s, we get:

$$\left(i\frac{\partial}{\partial t}-\omega_s(\mathbf{k})\right) a_s(\mathbf{k}, t) = \int d\mathbf{k}_1 d\mathbf{k}_2 N_{ss_1s_2}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\, \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\, a_{s_1}(\mathbf{k}_1)\, a_{s_2}(\mathbf{k}_2)\,,$$
$$N_{ss_1s_2} = \int_0^\infty d\tau_1 \int_0^\infty d\tau_2\, M_{ss_1s_2}(\tau_1, \tau_2, \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \exp\{i\omega_{s_1}(\mathbf{k}_1)\tau_1 + i\omega_{s_2}(\mathbf{k}_2)\tau_2\}\,. \qquad (1.13)$$

If the starting equation has the form

$$i\frac{\partial\chi_n}{\partial t} + \int H_{nm}(\mathbf{r}-\mathbf{r}')\,\chi_m(\mathbf{r}')\, d\mathbf{r}' = \iint H^{(1)}_{nml}(\mathbf{r}-\mathbf{r}', \mathbf{r}-\mathbf{r}'')\,\chi_m(\mathbf{r}')\,\chi_l(\mathbf{r}'')\, d\mathbf{r}'\, d\mathbf{r}''$$

then the equations at once assume the form (1. 13).

Note that no use is made of the commutative nature of the multiplication of the quantities χ and a_s. This means that the above procedure is applicable to quantum theory. In this case, equations (1. 13) and (1. 14) will be the Heisenberg equations for the operators of particle annihilation and production. We now note that $a_{-s}(k) = a_s^*(-k)$. Substituting this relation in (1. 13) and changing sign where necessary, we finally get:

$$\left(i\frac{\partial}{\partial t}-\omega_s(\mathbf{k})\right) a_s(\mathbf{k}) = \int d\mathbf{k}_1\, d\mathbf{k}_2\, [N^{(1)}_{s\,s_1s_2}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\,\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\, a_{s_1}(\mathbf{k}_1)\times$$
$$\times a_{s_2}(\mathbf{k}_2) + 2N^{(2)}_{s\,s_1s_2}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\,\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)\, a_{s_1}^*(\mathbf{k}_1)\, a_{s_2}(\mathbf{k}_2) + N^{(3)}_{s\,s_1s_2}\times \qquad (1.14)$$
$$\times(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\,\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\, a_{s_1}^*(\mathbf{k}_1)\, a_{s_2}^*(\mathbf{k}_2)]\,.$$

We shall call Eqs. (1. 14) the equations in normal form. They coincide in number with the number of wave modes.

Special interest attaches to the case in which the medium is transparent. Then all $\omega_s(k)$ are real, and there is a normalization of a_s such that definite symmetry relations exist between $N^{(1)}$, $N^{(2)}$, and $N^{(3)}$.

If we determine the transparency of the medium as the invariance of some real functional of a_s with a bilinear and a trilinear part, we can show that this functional H is a Hamiltonian for the system, and Eqs. (1. 14) are obtained by varying H in accordance with the rule

$$i\frac{\partial a_s}{\partial t} = \frac{\delta H}{\delta a_s^*}\,. \qquad (1.15)$$

The most general form of this Hamiltonian is

$$H = \sum_s \int \omega_s(\mathbf{k})\, a_s(\mathbf{k})\, a_s^*(\mathbf{k})\, d\mathbf{k} + \int [H^{(1)}_{s\,s_1s_2}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\, a_s^*(\mathbf{k})\, a_{s_1}(\mathbf{k}_1)\, a_{s_2}(\mathbf{k}_2) +$$
$$+ H^{*(1)}_{ss_1s_2}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\, a_s(\mathbf{k})\, a_{s_1}^*(\mathbf{k}_1)\, a_{s_1}^*(\mathbf{k}_2)]\,\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\, d\mathbf{k}_1\, d\mathbf{k}_2 +$$
$$+ \int [H^{(2)}_{ss_1s_2}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\, a_s(\mathbf{k})\, a_{s_1}(\mathbf{k}_1)\, a_{s_2}(\mathbf{k}_2) + \qquad (1.16)$$
$$+ H^{*(2)}_{s\,s_1s_2}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\, a_s^*(\mathbf{k})\, a_{s_1}^*(\mathbf{k}_1)\, a_{s_2}^*(\mathbf{k}_2)]\,\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\, d\mathbf{k}_1\, d\mathbf{k}_2$$

where

$$H^{(1)}_{s\,s_1s_2}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = H^{(1)}_{s\,s_2s_1}(\mathbf{k}, \mathbf{k}_2, \mathbf{k}_1)\,,$$
$$H^{(2)}_{s\,s_1s_2}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = H^{(2)}_{s_1\,ss_2}(\mathbf{k}_1, \mathbf{k}, \mathbf{k}_2) = H^{(2)}_{s\,s_2s_1}(\mathbf{k}, \mathbf{k}_2, \mathbf{k}_1)\,. \qquad (1.17)$$

We can now write down the most general form of the normal equations in a transparent medium

$$\left(i\frac{\partial}{\partial t}-\omega_s(\mathbf{k})\right) a_s(\mathbf{k}) = \int d\mathbf{k}_1\, d\mathbf{k}_2\, [H^{(1)}_{s\,s_1s_2}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\, a_{s_1}(\mathbf{k}_1)\, a_{s_2}(\mathbf{k}_2)\,\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2) +$$
$$+ 2H^*_{s_1ss_2}(\mathbf{k}_1, \mathbf{k}, \mathbf{k}_2)\,\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)\, a_{s_1}^*(\mathbf{k}_1)\, a_{s_2}(\mathbf{k}_2) +$$
$$+ H^{*(2)}_{s\,s_1s_2}\, a_{s_1}^*(\mathbf{k}_1)\, a_{s_2}^*(\mathbf{k}_2)\,\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)]\,.$$

Many problems of plasma turbulence lead to non-Hamiltonian equations, even if the free equations are Hamiltonian [6, 7]. In these circumstances an important part is played by so-called nonlinear Landau damping. However, we shall confine ourselves to the Hamiltonian case, and in constructing models we shall start directly from the Hamiltonians.

2. Three-wave model. Let us consider the following model. Suppose we have waves of three modes, A, B, and C, described by the variables a_k, b_k, and c_k, respectively. The laws of dispersion are as follows:

$$\omega_A \approx \omega_1, \qquad \omega_B = \omega_2, \qquad \omega_C = \sqrt{\omega_3^2 + s^2k^2} . \tag{2.1}$$

The parameter ε is small and subsequently tends to zero. The laws of dispersion are subject to the conditions:

$$\omega_3 < \omega_1 - \omega_2 \ll \omega_2 . \tag{2.2}$$

The model is considered in the region of wave numbers

$$|k| < s^{-1}\sqrt{\omega_2^2 - \omega_3^2} . \tag{2.3}$$

In this region the only first-order dynamic process of the theory of perturbations is

$$A \rightleftarrows B + C \tag{2.4}$$

which obeys the conservation laws

$$\omega_A = \omega_B + \omega_C, \qquad \mathbf{k}_A = \mathbf{k}_B + \mathbf{k}_C . \tag{2.5}$$

We select the Hamiltonian of the system in the form

$$\begin{aligned} H = \int \omega_A a_\mathbf{k} a_\mathbf{k}^* d\mathbf{k} + \int \omega_B b_\mathbf{k} b_\mathbf{k}^* d\mathbf{k} + \int \omega_C(\mathbf{k})\, c_\mathbf{k} c_\mathbf{k}^* d\mathbf{k} + \\ + \lambda \int (a_\mathbf{k}^* b_{\mathbf{k}_1} c_{\mathbf{k}_2} + a_\mathbf{k} b_{\mathbf{k}_1}^* c_{\mathbf{k}_2}^*)\, \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\, d\mathbf{k}_1\, d\mathbf{k}_2 . \end{aligned} \tag{2.6}$$

Clearly, the model is a classical analogue of the Lee quantum model. The absence of cross terms is unimportant, since in the region in question they do not contribute to the kinetic equation.

It would be possible to study a more complex model by introducing some form factor into the interaction Hamiltonian, but this would not affect the qualitative results.

The dynamic equations in normal form are obtained by varying Hamiltonian (2.6):

$$\begin{aligned} i\frac{\partial a_\mathbf{k}}{\partial t} - \omega_A a_\mathbf{k} &= \lambda \int b_{\mathbf{k}'} c_{\mathbf{k}''} \delta(\mathbf{k} - \mathbf{k}' - \mathbf{k}'')\, d\mathbf{k}'\, d\mathbf{k}'', \\ i\frac{\partial b_\mathbf{k}}{\partial t} - \omega_B b_\mathbf{k} &= \lambda \int a_{\mathbf{k}'} c_{\mathbf{k}''}^* \delta(\mathbf{k}' - \mathbf{k} - \mathbf{k}'')\, d\mathbf{k}'\, d\mathbf{k}'', \\ i\frac{\partial c_\mathbf{k}}{\partial t} - \omega_C c_\mathbf{k} &= \lambda \int b_{\mathbf{k}'}^* a_{\mathbf{k}''} \delta(\mathbf{k}'' - \mathbf{k}' - \mathbf{k})\, d\mathbf{k}'\, d\mathbf{k}'' . \end{aligned} \tag{2.7}$$

From this dynamic system we can obtain a system of kinetic equations, as Galeev and Karpman did in [1],

$$\begin{aligned} \frac{\partial A_\mathbf{k}}{\partial t} &= 4\pi\lambda^2 \int \delta(\mathbf{k} - \mathbf{k}' - \mathbf{k}'')\, \delta(\omega_1 - \omega_2 - \omega_3)(\mathbf{k}'')) \times \\ &\qquad \times (B_{\mathbf{k}'} C_{\mathbf{k}''} - A_\mathbf{k} B_{\mathbf{k}'} - A_\mathbf{k} C_{\mathbf{k}''})\, d\mathbf{k}'\, d\mathbf{k}'' + \nu_\mathbf{k} A_\mathbf{k}, \\ \frac{\partial B_\mathbf{k}}{\partial t} &= 4\pi^2\lambda \int \delta(\mathbf{k}' - \mathbf{k} - \mathbf{k}'')\, \delta(\omega_1 - \omega_2 - \omega_3(\mathbf{k}'')) \times \\ &\qquad \times (A_{\mathbf{k}'} C_{\mathbf{k}''} + A_{\mathbf{k}'} B_\mathbf{k} - B_\mathbf{k} C_{\mathbf{k}''})\, d\mathbf{k}'\, d\mathbf{k}'', \\ \frac{\partial C_\mathbf{k}}{\partial t} &= 4\pi^2\lambda \int \delta(\mathbf{k}'' - \mathbf{k}' - \mathbf{k})\, \delta(\omega_1 - \omega_2(\mathbf{k}'') - \omega_3(\mathbf{k})) \times \\ &\qquad \times (A_{\mathbf{k}'} B_{\mathbf{k}''} + C_\mathbf{k} A_{\mathbf{k}'} - C_\mathbf{k} B_{\mathbf{k}''})\, d\mathbf{k}'\, d\mathbf{k}'', \end{aligned} \tag{2.8}$$

where

$$A_\mathbf{k} = \langle a_\mathbf{k} a_\mathbf{k}^* \rangle, \qquad B_\mathbf{k} = \langle b_\mathbf{k} b_\mathbf{k}^* \rangle, \qquad C_\mathbf{k} = \langle c_\mathbf{k} c_\mathbf{k}^* \rangle ,$$

are the mean squares of the wave amplitudes. Obviously, Eqs. (2.8) are satisfied by the Rayleigh-Jeans solutions:

$$A_\mathbf{k} = T/\omega_A, \qquad B_\mathbf{k} = T/\omega_B, \qquad C_\mathbf{k} = T/\omega_C .$$

In the first two equations (2.8) we neglect the term εk^2. Moreover, the term ν_k, which takes into account the sources of the A-waves, is formally introduced.

Conservation laws (2. 5) imply that only those C-waves whose wave numbers satisfy the condition

$$k_c^2 = k_0^2 - \varepsilon \frac{k_A^2}{\omega_{s^2}}, \qquad k_0^2 = \frac{(\omega_1 - \omega_2)^2 - \omega_3^2}{s^2}, \tag{2. 9}$$

i. e., lie in a very narrow spherical layer, will take part in the process.

On integrating the last of equations (2. 8) with respect to k, we note that the term with the time derivative

$$\int \frac{\partial C}{\partial t} d\mathbf{k}$$

is of the order of εk^2 and may be neglected. Introducing the mean value $C = \langle C_{\mathbf{k}} \rangle$, where the averaging is carried out over the volume of the spherical layer, and assuming that the C-wave distribution is spherically symmetrical, we get

$$\int \delta(\mathbf{k} + \mathbf{k}' - \mathbf{k}'') (A_{\mathbf{k}'} B_{\mathbf{k}''} + C A_{\mathbf{k}'} - C B_{\mathbf{k}''}) d\mathbf{k}' \, d\mathbf{k}'' . \tag{2. 10}$$

We shall further confine ourselves to states for which

$$\frac{A_{\mathbf{k}'}}{A_{\mathbf{k}}} \ll \frac{1}{k_0}, \qquad \frac{B_{\mathbf{k}}'}{B_{\mathbf{k}}} \ll \frac{1}{k_0} .$$

We expand the right sides of (2. 9) in series with respect to k_0 and exclude from consideration everything beyond the second term. Terms containing the first power of k_0 vanish. Considering only stationary states, we get

$$\begin{aligned} (C - A_{\mathbf{k}})(B_{\mathbf{k}} + {}^1/_2\, k_0^2 \Delta B_{\mathbf{k}}) - C A_{\mathbf{k}} + \gamma_{\mathbf{k}} A_{\mathbf{k}} &= 0 \\ (C + B_{\mathbf{k}})(A_{\mathbf{k}} + {}^1/_2\, k_0^2 \Delta A_{\mathbf{k}}) - C B_{\mathbf{k}} &= 0 \end{aligned} \quad \left(\gamma_{\mathbf{k}} = \frac{\nu_{\mathbf{k}} \sqrt{\omega_3^2 + s^2 k_0^2}}{4\pi\lambda^2 s^2} \right) . \tag{2. 11}$$

Applying conditions (2. 10) to Eqs. (2. 11) we get:

$$\int A_{\mathbf{k}} \nu_{\mathbf{k}} d\mathbf{k} = 0 . \tag{2. 12}$$

This condition supplements system (2. 11). Further we put:

$$\nu_k = \nu_1 \quad \text{for } |\mathbf{k}| > k_1, \qquad \nu_k = 0 \quad \text{for } k_2 > |\mathbf{k}| > k_1, \qquad \nu_k = -\nu_2 \quad \text{for } |\mathbf{k}| > k_2 . \tag{2. 13}$$

The corresponding values of γ_k are γ_1 and γ_2.

This choice of sources simulates the real situation for instability in the longwave region and damping in the shortwave region. We shall assume that $k_2 \gg k_1$.

We now isolate B_k from the second of equations (2. 11) and, making use of the smallness of the term with the Laplacian, expand it in series with respect to k_0^2 up to the first term. Substituting the result in the first equation, we get the following equation for A_k:

$$\Delta A_{\mathbf{k}} + \frac{1}{C - A_{\mathbf{k}}} (\nabla A_{\mathbf{k}})^2 + \frac{\gamma_{\mathbf{k}} A_{\mathbf{k}} (c - Ak)}{k_0^2 C^2} = 0 . \tag{2. 14}$$

We shall solve it in three regions where the ν_k, and hence the γ_k, are constant.

1. Solution in the region of instability. We shall assume that the region of instability is narrow. Then the solution in this region can be obtained by expanding A_k in powers of k. We get (A_0 is still an unknown constant)

$$A_{\mathbf{k}} = A_0 - ak^2 \qquad \left(a = \frac{\gamma_1 A_0 (C - A_0)}{6 k_0^2 C^2} \right) . \tag{2. 15}$$

The condition of applicability of the expansion will be $a \ll 1$.

2. Solution in the region of transparency. In this region

$$\Delta A_k + \frac{1}{C - A_k} (\nabla A_k)^2 = 0 . \tag{2. 16}$$

The general spherically symmetric solution of (2. 15) will be

$$A_k = C + \alpha e^{\beta/k} \qquad (a, \beta = \text{const}). \tag{2. 17}$$

3. Solution in the region of damping. We shall assume that in this region A_k is small and cite the linearized equation (2. 13) and its solution:

$$\Delta A_k = \frac{\gamma_6}{Ck_0^2} A_k = 0, \qquad A_k = \frac{\delta}{k} \exp\left[-\left(\frac{\gamma_2}{C}\right)^{1/2} \frac{k_2}{k_0}\right] \qquad (\delta = \mathrm{const}) \tag{2. 18}$$

For the applicability of the starting assumptions it is necessary that

$$\gamma_2 / C \ll 1 . \tag{2. 19}$$

Correct to higher-order terms, condition (2. 12) gives

$$\frac{\gamma_1}{3} k_1^3 A_0 = \delta k_2 k_0 \sqrt{C\gamma^2} \exp\left[-\left(\frac{\gamma_2}{C}\right)^{1/2} \frac{k_2}{k_0}\right] . \tag{2. 20}$$

Matching the solutions at the point k_1, for the region of transparency we get the approximation

$$A_k \approx C - (C - A_0) \exp\left(-\frac{\gamma_1 A_0 k_1^3}{3C^2 k_0^2 k}\right). \tag{2. 21}$$

Note that A_k must be of the order of or less than C (in the long run this is justified). Then, as a result of (2. 15) the exponent is small, and A_k changes little over the region of transparency and is approximately equal to A_0. Therefore in matching at the point k_2 it is necessary to proceed with caution, since precisely in the neighborhood of k_2 there is a region where the linearization of Eq. (2. 14) is not valid and solution (2. 18) will be incorrect. Therefore at the point k_2 we shall match only the functions and not their derivatives. We have

$$A_0 \approx \frac{\delta}{k_2} \exp\left[-\left(\frac{\gamma_2}{c}\right)^{1/2} \frac{k_2}{k_0}\right] . \tag{2. 22}$$

Substituting (2. 22) in (2. 21), we get

$$C = \frac{1}{\gamma^2} \left(\frac{\gamma_1 k_1^3}{3k_0 k_2^2}\right)^2 . \tag{2. 23}$$

Now A_0 can be found from the following consideration. From (2. 21) it is clear that in the region of transparency A_k differs little from the Rayleigh-Jeans distribution for some unknown temperature. Assuming that the gas of the A-waves and the gas of the C-waves are in a state close to thermal equilibrium, we can put:

$$A_0 = \frac{\omega_C(k_0)}{\omega_A} C, \quad \omega_C(k_0) = \omega_1 - \omega_2, \quad \text{or} \quad A_0 = \frac{(\omega_1 - \omega_2)^2}{4\pi\lambda^2 s^2} \frac{1}{\nu_2} \left(\frac{\nu_1 k_1^3}{3k_0 k_2^2}\right)^2 . \tag{2. 24}$$

From (2. 24) it follows that $A_0 \ll C$. Substituting (2. 23) in (2. 15) and (2. 19), we get the corresponding conditions of applicability:

$$\frac{\gamma_2}{\gamma_1} \left(\frac{k_2}{k_1}\right)^4 \ll 1, \qquad \frac{\gamma_2}{\gamma_1} \frac{k_0 k_2^2}{k_1^3} \ll 1 . \tag{2. 25}$$

As may be seen from our work, the spectrum of weak turbulence really has the character usually attributed to it, i. e. , turbulence is established at the expense of the energy balance between waves generated in the region of instability and damped in the region of absorption. However, the simple estimate that equates the orders of the term $\nu_i A$ and the nonlinear term is very rough. It is interesting to note that in the region of transparency the spectrum differs little from the Rayleigh-Jeans distribution.

The method described can be generalized to include the class of problems for which the surfaces in k-space described by the equations

$$\omega(\mathbf{k} + \mathbf{k}_1) = \omega(\mathbf{k}) = \omega(\mathbf{k}_1), \qquad \omega(\mathbf{k} - \mathbf{k}_1) = \omega(\mathbf{k}) + \omega(\mathbf{k}_1)$$

is closed, while their maximum dimension is independent of k.

REFERENCES

1. A. A. Galeev and V. I. Karpman "Turbulent theory of a weakly nonequilibrium rarefied plasma and the structure of shock waves," ZhTEF, vol 44, p. 592.

2. M. Camac, "Shock-waves in collision-free plasmas," Nuclear Fusion, Supplement; Part 2, International Atomic Energy Agency, Vienna, 1962.

3. A. A. Vedenov, "Quasilinear theory of plasma oscillations," Collection: Problems of Plasma Theory [in Russian], vol. 3, Gosatomizdat, 1963.

4. B. B. Kadomtsev and V. I. Petviashvili, "Weakly turbulent plasma in a magnetic field," ZhETF, vol. 43, p. 2234, 1963.

5. V. I. Karpman, "Nonlinear effects in the electrodynamics of a transparent medium," ZhTEF, vol. 44, p. 1307, 1963.

6. V. I. Karpman, "On the theory of a weakly turbulent plasma," DAN SSSR, vol. 152, no. 2, p. 587, 1963.

7. V. I. Petviashvili, "Nonlinear oscillations and some effects due to a longitudinal current in a plasma," ZhTEF, vol. 45, p. 1467, 1963.

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